MAC 2311
Notes
Section 4.4
Indeterminant Forms

L'Hospital's Rule   type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

note: L'Hospital's Rule can be repeated multiple times

Examples

① $\lim_{x \to -2} \frac{x^3+8}{x+2}$ current form $\frac{0}{0}$

$\overset{L}{=} \lim_{x \to -2} \frac{3x^2}{1} = 3(-2)^2 = \boxed{12}$

② $\lim_{x \to \infty} \frac{\ln \sqrt{x}}{x^2}$ form $\frac{\infty}{\infty}$

$\overset{L}{=} \lim_{x \to \infty} \frac{\frac{1}{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}}}{2x}$

$= \lim_{x \to \infty} \frac{1}{2x} \cdot \frac{1}{x^{1/2}} \cdot \frac{1}{2} \cdot \frac{1}{x^{1/2}}$

$= \lim_{x \to \infty} \frac{1}{4x^2} = \boxed{0}$

Format	$0 \cdot \infty$

the answer may or may not be zero

strategy

$$a \cdot b = \frac{a}{\frac{1}{b}}$$

This converts the form to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example	(#44)

$$\boxed{\lim_{x \to \infty} \sqrt{x}\, e^{-x/2}}$$

form	$0 \cdot \infty$

$$\lim_{x \to \infty} \frac{x^{1/2}}{e^{x/2}} \quad \text{now } \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x \to \infty} \frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}}$$

$$= \lim_{x \to \infty} \frac{1}{x^{1/2}e^{x/2}} \Rightarrow \boxed{0}$$

Type $\infty - \infty$

note that $\infty - \infty$ may not be equal to zero.

The format

$$\boxed{\lim_{x \to a} [f(x) - g(x)]}$$

The strategy (fractions)

1. Find an lcd for the two terms.
2. This will convert the expression into the format $\frac{0}{0}$ or $\frac{\infty}{\infty}$
3. Now use L'Hospital's rule to finish the analysis

Example (#51)

$$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$$

$\boxed{\text{format } \infty - \infty}$

$$= \lim_{x \to 1} \left(\frac{x(\ln x) - 1(x-1)}{(\ln x)(x-1)} \right)$$

$$= \lim_{x \to 1} \frac{x \ln x - x + 1}{(\ln x)(x-1)}$$

now $\frac{0}{0}$

$$\overset{L}{=} \lim_{x \to 1} \frac{1(\ln x) + x\left(\frac{1}{x}\right) - 1}{\left(\frac{1}{x}\right)(x-1) + \ln x}$$

$$= \lim_{x \to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} \qquad \boxed{\text{form}} \quad \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{x^{-2} + \frac{1}{x}} \qquad \underline{\text{note}} \quad \frac{-1}{x} = -1x^{-1}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

The strategy	non-fractions $\Rightarrow$ use conjugates or factoring

Example $\lim_{x \to \infty} e^x - x$ $\boxed{\text{form}}$ $\infty - \infty$

conjugates

$$\lim_{x \to \infty} \frac{(e^x - x) \cdot (e^x + x)}{(e^x + x)}$$

$$\lim_{x \to \infty} \frac{e^{2x} - x^2}{e^x + x} \leftarrow \frac{\infty - \infty}{\infty + \infty}$$

same problem

$\Rightarrow$ need to break it down further

$$\lim_{x\to\infty} \frac{e^{2x} - x^2}{e^x + x}$$

logically e^{2x} will grow faster than e^x $\Rightarrow \infty$

$$\lim_{x\to\infty} \left(\frac{e^{2x}}{e^x + x} - \frac{x^2}{e^x + x} \right)$$

we need to prove this.

part 1

$$\lim_{x\to\infty} \frac{e^{2x}}{e^x + x} \qquad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{2e^{2x}}{e^x + 1} \qquad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{4e^{2x}}{e^x}$$

$$= \lim_{x\to\infty} \frac{4e^x \cancel{e^x}}{\cancel{e^x}} = \infty$$

part 2

$$\lim_{x\to\infty} \frac{-x^2}{e^x + x} \qquad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{-2x}{e^x + 1} \qquad \frac{\infty}{\infty}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{-2}{e^x + 1} = \frac{-2}{\infty} = 0$$

so the total limit is $\infty + 0 = \boxed{\infty}$

Format Indeterminate Powers

$$\boxed{\lim_{x \to a} f(x)^{g(x)}} \quad \boxed{0^0,\ \infty^0,\ 1^\infty}$$

Strategy Use log rules
Recall
$$a^x = e^{x \ln a}$$

This is a similar strategy to logarithmic differentiation.

Examples

1. (#58)

$$\lim_{x \to 0^+} (\tan 2x)^x \qquad (\text{form: } 0^0)$$

$$y = \lim_{x \to 0^+} (\tan 2x)^x$$

$$y = \lim_{x \to 0^+} e^{x \ln(\tan 2x)}$$

$$\ln y = \lim_{x \to 0^+} \ln\left(e^{x \ln(\tan 2x)}\right)$$

$$\ln y = \lim_{x \to 0^+} x \ln(\tan 2x)$$

$$\ln y = \lim_{x \to 0^+} \underbrace{x(\ln(\tan 2x))}_{\text{form } 0 \cdot \infty}$$

note $0 \cdot \infty$ is not necessarily zero

to work with $0 \cdot \infty$, convert it to complex fraction which changes the form to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (L'Hospital)

$$\ln y = \lim_{x \to 0^+} \frac{\ln(\tan 2x)}{\frac{1}{x}}$$

$$\ln y = \lim_{x \to 0^+} \frac{\frac{1}{\tan 2x} \cdot \sec^2 2x \cdot 2}{\frac{-1}{x^2}}$$

$$\ln y = \lim_{x \to 0^+} \frac{\cos 2x}{\sin 2x} \cdot \frac{1}{(\cos 2x)^2} \cdot 2 \cdot \frac{-x^2}{1}$$

$$\ln y = \lim_{x \to 0^+} \frac{-2x^2}{\sin 2x \cos 2x}$$

$$\ln y = \lim_{x \to 0^+} \frac{-2x^2}{\frac{1}{2}\sin 4x} \quad \text{form } \frac{0}{0}$$

$$\ln y \overset{L}{=} \lim_{x \to 0^+} \frac{-4x}{\frac{1}{2}\cos 4x \cdot 4} = \frac{0}{2} = 0$$

$$y = e^0 = \boxed{1}$$

Summary

Indeterminate Forms

$\frac{\infty}{\infty}$ or $\frac{0}{0}$ — L'Hospital's Rule

$\infty - \infty$ — lcd, factoring or conjugates to change to $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$0 \cdot \infty$ — convert using $a \cdot b = \frac{a}{\frac{1}{b}}$ which is now $\frac{\infty}{\infty}$ or $\frac{0}{0}$

0^0, ∞^0, 1^∞ — use $a^x = e^{x \ln a}$ to start. You may need to use other techniques to finish.